A study of the structure of the magnetohydrodynamic switch-on shock in steady plane motion

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The structure of the steady magnetohydrodynamic switch-on shock wave is investigated for several orderings of the four diffusivities involved in the problem. The various orderings are approximated to by allowing one or more of the appropriate diffusivities to approach zero, and approximate solutions that are uniformly valid to order unity are sought. In general, singular perturbation problems are encountered, the number occurring (from zero to a maximum of three) depending upon the ordering of the diffusivities and the magnitude of the downstream velocity normal to the shock relative to certain critical velocities downstream of the shock. Where necessary, the approximate solutions are rendered uniformly valid to first order by the insertion of boundary layers, for which the approximate equations are determined to first order. For most of the cases considered, the limiting forms of the integral curves are determined and they are sketched in appropriate three-dimensional phase spaces.

1. Introduction

The study of the structure of steady magnetohydrodynamic shock waves in plane motion using continuum theory was initiated by Marshall (1956). He studied the 'parallel' shock, i.e. a case where the flow is normal and the magnetic field is parallel to the shock. He obtained explicit results for the shock structure for two limiting cases: (i) magnetic diffusivity large (low electrical conductivity), and (ii) magnetic diffusivity small (high electrical conductivity) compared with the other diffusivities. The parallel shock for the case of low electrical conductivity was also studied by Burgers (1957) and Whitham (1959). Recently, Ludford (1959) studied, in addition to the parallel shock, some general features of switch-on, switch-off, and transverse magnetohydrodynamic shocks, as defined by Friedrichs (1957). His treatment is essentially restricted to moderate to low electrical conductivity. Independently, Bleviss (1959) investigated in detail the structure of the switch-on shock for the case of low electrical conductivity.

This paper presents a more general investigation of the structure of the switch-on shock. The switch-on shock is a shock that has the velocity and magnetic field vectors normal to it on the upstream side and oblique to it on the downstream side (see figure 1). The name derives from the fact that tangential components of velocity and magnetic field are 'switched-on'. This turning of the normal flow, which is not possible with hydrodynamic shocks, is accomplished here by Maxwell

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stresses that are parallel to the shock front. This shock is of interest because of its peculiar properties and because it is generally more complex than the parallel shock.

The equations that govern the shock structure can be reduced to a system of four first-order non-linear ordinary differential equations. In this system of equations, each of the four different derivatives is multiplied by a different one of the four diffusivities. In general, this system of equations must be solved

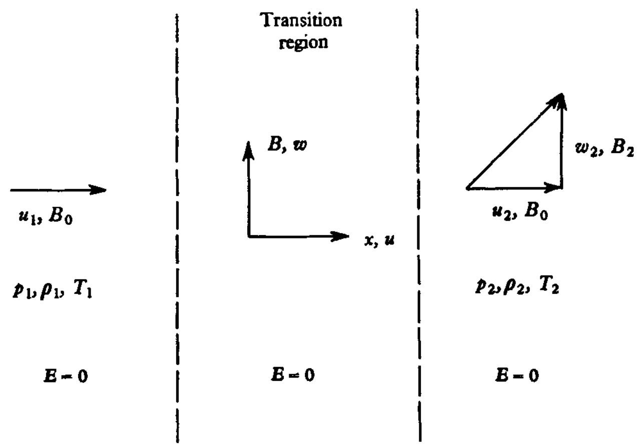

FIGURE 1. Switch-on shock wave.

numerically. When order-of-magnitude differences occur among the diffusivities, the system of equations can be approximated to by allowing the appropriate diffusivities to approach zero. These limiting cases are, of course, simpler and many of them can be solved analytically. A few of the many possible order-of-magnitude orderings of the diffusivities represent physically realistic cases, the other cases being only of mathematical interest.

In this paper, several such limiting cases are investigated and approximate solutions that are uniformly valid to order unity are sought. In general, 'singular perturbation' or 'boundary-layer type' problems are encountered and they are investigated by studying the limiting forms of the integral curves in phase space. The conditions for the appearance and location of the boundary layers and the boundary-layer equations to first order are determined.

A discussion of the validity of some of the main assumptions and of the necessary (but not sufficient) conditions for the existence of a switch-on shock has been given by Bleviss (1959) and is not repeated here. The sufficiency conditions, which must depend upon the boundary conditions, are not known and will not be discussed, but a paper by Cole (1959) sheds some light on this subject.

2. Discussion of equations of motion

The steady, plane, continuum flow of a compressible, viscous, heat conducting, electrically conducting, electrically neutral, perfect gas is considered. The electrical conductivity is assumed to be a scalar. The equations of motion for the switch-on shock will not be derived here since they have already been given by Bleviss (1959), or can be readily deduced from a convenient form of the general equations of steady magnetohydrodynamics as given by Bleviss (1958). A system of four first-order non-linear ordinary differential equations in the four dependent variables B, w, u and T is obtained by eliminating the variables p, ρ and h through the use of the continuity equation $\rho u = m =$ constant and the perfect gas relations $p = \rho RT$ and $h = C_p T$, where B is the magnetic field component parallel to the shock, w and u are the velocity components parallel and normal, respectively, and h is specific enthalpy. It is readily deduced from the general equations that the x-component of the magnetic field is a constant (denoted here by B_0) and that the electric field is zero.

Using the notation and co-ordinate system shown in figure 1 and the rationalized M.K.S.Q. system of units, the equations of motion can be written in the following convenient form:

$$\lambda \frac{dB}{dx} = uB - wB_0 = (u-u_2)B + (B-B_2)u_2 - (w-w_2)B_0 = J(B,w,u), \quad (1a)$$

$$\eta \frac{dw}{dx} = mw - \frac{BB_0}{\mu} = m(w-w_2) - (B-B_2)\frac{B_0}{\mu} = L(B,w), \quad (1b)$$

$$\eta'' u \frac{du}{dx} = mR(T-T_2) + m\left(u - \frac{RT_2}{u_2}\right)(u-u_2) + \frac{B^2-B_2^2}{2\mu}u = M(B,u,T), \quad (1c)$$

$$k\frac{dT}{dx} = mC_v(T-T_2) + \frac{mRT_2}{u_2}(u-u_2) - \frac{m}{2}(u-u_2)^2 - \frac{m}{2}(w-w_2)^2 - \frac{B^2-B_2^2}{2\mu}u + \frac{B_0}{\mu}(B-B_2)w = N(B,w,u,T). \quad (1d)$$

The coefficients describing physical properties of the fluid are shear coefficient of viscosity η, longitudinal coefficient of viscosity η'', coefficient of thermal conduction k, magnetic permeability μ, electrical conductivity σ, magnetic diffusivity $\lambda = 1/\mu\sigma$, and specific heat at constant volume C_v. Equation (1a) is a combination of Ohm's law and Maxwell's current equation; (1b) is the once-integrated tangential momentum equation with the constant of integration equal to zero since $w = B = 0$ at 1 (the uniform flow region upstream of the shock, i.e. $x = -\infty$). Equations (1c) and (1d) are the longitudinal momentum and the energy equations, respectively, integrated once and with constants of integration evaluated at 2, the uniform flow region downstream of the shock ($x = +\infty$). It should be noted that it has been implicitly assumed in these equations (and in the shock relations) that the current is free to flow in closed circles. Thus, the present one-dimensional problem must be viewed as an approximation to an axisymmetrical problem at a large distance from the axis of symmetry.

To avoid unnecessary complications in notation and equations, system (1) will be retained throughout this paper in its dimensional form, as written above. In addition, it is convenient to refer to all of the parameters λ, η, η'', and k as diffusivities in the various order-of-magnitude statements, even though the actual diffusivities are λ, η/ρ, η''/ρ, and $k/\rho C_p$. Thus, η, η'', $k \ll \lambda$ should be interpreted as an order-of-magnitude statement about the corresponding diffusivities.

The switch-on shock relations are obtained by setting the left-hand side of system (1) equal to zero at 1 and/or 2. The following simple shock relations obtained from (1*a*) and (1*b*) are listed here for future reference:

$$u_2 B_2 = w_2 B_0, \tag{2a}$$

$$m w_2 = \frac{B_0 B_2}{\mu}, \tag{2b}$$

$$u_2 = \frac{B_0^2}{\mu m}, \tag{2c}$$

$$w_2^2 = \frac{B_2^2}{\mu \rho_2} \equiv b_2^2. \tag{2d}$$

Equations (2*a*) and (2*b*) were used to obtain the second forms of the right-hand sides of (1*a*) and (1*b*).

Solutions of system (1) may be represented as integral curves between the singularities at 1 and 2 in the (B, w, u, T) phase space with x as a parameter along the curves. In general, this system of equations must be solved numerically using methods similar to those of Gilbarg & Paolucci (1953) and Marshall (1956).

When order-of-magnitude differences occur among the diffusivities, the system of equations and, hence, the solution can be approximated to by allowing the appropriate diffusivities to approach zero. Such limiting cases are simpler and many of them can be solved analytically. For example, if $\eta, \eta'', k \ll \lambda$ one can set the left-hand sides of (1*b*), (1*c*) and (1*d*) equal to zero (i.e. $\eta = \eta'' = k = 0$) and the resulting system of equations is readily solved analytically, the approximate solution being uniformly valid to order unity for a certain range of conditions. Of the many possible order-of-magnitude orderings of the four diffusivities, a few represent physically realistic cases, the others being primarily of mathematical interest for the system (1). In general, 'singular perturbation' or 'boundary-layer type' problems arise in such limiting cases, these problems being heralded by double-valued shock profiles.

The limiting cases corresponding to several order-of-magnitude orderings will be investigated in detail in this paper. Approximate solutions that are uniformly valid to first order will be sought. A very convenient non-standard method for handling the singular perturbation problems for any limiting case will be presented. The method is based upon a study of the limiting forms of the integral curves in the phase space.

Order-of-magnitude orderings that correspond to physically realistic possibilities will now be discussed. Within the assumptions of this paper, the diffusivities η and η'' should be of the same order of magnitude.† A natural ordering arises in highly ionized gases because the Prandtl number becomes small, i.e. $\eta, \eta'' \ll k$. The magnetic diffusivity will be large compared with the others over a large range of hypersonic flow conditions. This leads to the physically realistic cases $\eta, \eta'', k \ll \lambda$ and $\eta, \eta'' \ll k \ll \lambda$. At high enough temperatures, λ can be of the

† If the gas were highly ionized, the density were low enough, and the magnetic field were strong enough, η and η'' could differ considerably but, under such conditions, k and σ would have to be tensors.

same order or much less than the other diffusivities. This leads to the possibilities $\eta, \eta'' \ll k, \lambda$; $\eta, \eta'' \ll \lambda \ll k$; $\eta, \eta'', \lambda \ll k$; and $\lambda \ll \eta, \eta'' \ll k$.

It is useful to write out two reduced systems of equations obtained from system (1) by, first, setting $\eta = 0$ and eliminating w from the other equations and, secondly, setting $\lambda = 0$ and eliminating B from the other equations. The notation $\eta = 0$ (or $\lambda = 0$) is just a convenient way of making the mathematical statement that the left-hand side of (1*b*) [or (1*a*)] is set equal to zero. For $\eta = 0$, the reduced system is

$$\lambda \frac{dB}{dx} = (u-u_2)B, \tag{3a}$$

$$\eta'' u \frac{du}{dx} = mR(T-T_2) + m\left(u - \frac{RT_2}{u_2}\right)(u-u_2) + \frac{B^2 - B_2^2}{2\mu} u, \tag{3b}$$

$$k\frac{dT}{dx} = mC_v(T-T_2) - (u-u_2)\left[\frac{B^2-B_2^2}{2\mu} + \frac{m}{2}(u-u_2) - \frac{mRT_2}{u_2}\right], \tag{3c}$$

with w given by

$$w = u_2 \frac{B}{B_0}. \tag{3d}$$

For $\lambda = 0$, the reduced system is

$$\eta \frac{u}{m}\frac{dw}{dx} = (u-u_2)w, \tag{4a}$$

$$\eta'' \frac{u^2}{m}\frac{du}{dx} = R(T-T_2)u + u\left(u - \frac{RT_2}{u_2}\right)(u-u_2) + \frac{u_2}{2}\left(w^2 - \frac{u^2}{u_2^2}w_2^2\right), \tag{4b}$$

$$k\frac{u}{m}\frac{dT}{dx} = C_v(T-T_2)u - \frac{u-u_2}{2}\left[w^2 - \frac{u}{u_2}w_2^2 + u(u-u_2) - \frac{2RT_2 u}{u_2}\right], \tag{4c}$$

with B given by

$$B = \frac{w}{u}B_0. \tag{4d}$$

Equations (2) have been used to obtain these forms. The limiting integral curves in these three-dimensional phase spaces will be shown later. For the moment, these reduced systems are useful to illustrate in a simple way the occurrence of discontinuities or boundary layers.

To illustrate the difficulties encountered, consider the case $\eta, \lambda \ll \eta'', k$. The standard procedure for obtaining the lowest order or first approximation is to set $\eta = \lambda = 0$ in system (1) and study the consequences. This also corresponds to setting $\lambda = 0$ in system (3) or $\eta = 0$ in (4). Consider system (3). When $\lambda = 0$, at least one of the quantities B and $u - u_2$ is zero. Since the boundary condition at 1 ($x = -\infty$) is $B = B_1 = 0$ (i.e. no tangential component of magnetic field), the integral curve from 1 to 2 must leave 1 in the plane $B = 0$ and must lie in that plane until $u = u_2$ and then the curve must lie in the plane $u = u_2$ until 2 ($x = +\infty$) is reached, unless difficulties arise. Equations (3*b*) and (3*c*) with $B = 0$ are the equations for a hydrodynamic shock between the upstream end-point 1 and a downstream end-point 3 that is different from 2 (it can be shown that $T_3 \geqslant T_2$ and $u_3 \leqslant u_2$). Hence, the integral curve leaves 1 along *some* integral curve of a hydrodynamic shock. In the plane $u = u_2$, the integral curve would have to satisfy the equations [(3b) and (3c) with $u = u_2$]

$$mR(T-T_2)+\frac{B^2-B_2^2}{2\mu}u_2 = 0,$$

$$k\frac{dT}{dx} = mC_v(T-T_2).$$

The second equation shows that T reaches its downstream value T_2 at $x = -\infty$. Since the upstream value T_1 also occurs at $x = -\infty$, this would mean that the temperature profile is double valued. The other quantities of interest will also exhibit double-valued profiles [cf. figure $D\,5d$ in Hayes (1958) and figure 4 in Bleviss (1959)]. In order to recover single-valued profiles and satisfy the boundary conditions at $x = +\infty$ one needs to insert a discontinuity for $\eta = \lambda = 0$. To obtain a uniformly valid approximation for η and λ near zero this discontinuity must be replaced by a thin transition layer (referred to hereinafter as a boundary layer).

Once it has been determined that a discontinuity or boundary layer is necessary, a standard method can be followed for determining the location of the boundary layer and the first-order approximate equations that determine its structure. Roughly and briefly, the method is as follows. One assumes a location for the boundary layer and writes the variables in appropriate form relative to this location. Then one divides each of the variables, dependent and independent, by a different unknown scale factor, each factor being the parameter that goes to zero to some, as yet undetermined, positive exponent. A scale factor describes how the corresponding variable behaves in the boundary layer as the parameter goes to zero. Therefore, the variable divided by the scale factor is an $O(1)$ quantity in the boundary layer. One then substitutes these $O(1)$ variables into the full set of equations, makes assumptions about the exponents, obtains reduced sets of equations to first order as the parameter goes to zero, and sees if any of the solutions to these sets of equations is sensible and consistent with the boundary conditions for the discontinuity. If proper equations cannot be found a different location is assumed and the procedure is repeated. In the process, both the equations and the scale factors are determined. If, as in the present case, there are several variables, it is clear that the method can be tedious. Often, the procedure can be considerably shortened by judicious guessing as to the location of the boundary layer and which terms in the equations should be retained.

In the present problem there is a much more convenient and direct method for determining the boundary-layer equations. It turns out that the discontinuity or boundary layer is always located at the downstream side of the switch-on shock. This result was assumed from previous experience and then proved by its success in all cases studied. This is the reason why all of the equations have been written in terms of the end-point 2. By studying the behaviour of the integral curves in the neighbourhood of 2 and requiring that these curves approach 2 as $x \to +\infty$, the conditions for the appearance of the boundary layers, the scale factors, and, hence, the boundary-layer equations to first order are readily determined. With this information, the remaining portions of the integral curves are easily deduced.

3. Behaviour of the intergral curve near 2

Following the usual procedure for studying the behaviour of the integral curve in the neighbourhood of a point, the system (1) is linearized near 2 by writing $B = B_2 + \Delta B$, etc., and dropping higher-order terms. The solution of this system is assumed to be of the form

$$B - B_2 \equiv \Delta B = Ce^{\kappa x},$$
$$w - w_2 \equiv \Delta w = C' e^{\kappa x},$$
$$\text{etc.}$$

A solution is possible only if κ satisfies the quartic

$$[L_B J_w - (J_B - \lambda\kappa)(L_w - \eta\kappa)][(N_T - k\kappa)(M_u - \eta'' u_2 \kappa) - N_u M_T] + J_u M_B (N_T - k\kappa)(L_w - \eta\kappa) = 0, \quad (5)$$

where

$$\left.\begin{aligned}
&J_B = \left(\frac{\partial J}{\partial B}\right)_2 = u_2, && J_w = -B_0, && J_u = B_2, && J_T = 0;\\
&L_B = \left(\frac{\partial L}{\partial B}\right)_2 = -\frac{B_0}{\mu}, && L_w = m, && L_u = 0, && L_T = 0;\\
&M_B = \frac{u_2 B_2}{\mu}, && M_w = 0, && M_u = \frac{m}{u_2}\left(u_2^2 - \frac{a_2^2}{\gamma}\right), && M_T = mR;\\
&N_B = \frac{B_0 w_2}{\mu} - \frac{u_2 B_2}{\mu} = 0, && N_w = 0, && N_u = \frac{m a_2^2}{\gamma u_2}, && N_T = mC_v.
\end{aligned}\right\} \quad (6)$$

Other relations of interest are

$$\left.\begin{aligned}
&L_B J_w - J_B L_w = \frac{B_0^2}{\mu} - m u_2 = 0,\\
&N_T M_u - N_u M_T = \frac{m^2 C_v}{u_2}(u_2^2 - a_2^2),\\
&J_B(N_T M_u - N_u M_T) - J_u M_B N_T = m^2 C_v (u_2^2 - a_2^2 - b_2^2),\\
&J_B M_u - J_u M_B = m\left(u_2^2 - \frac{a_2^2}{\gamma} - b_2^2\right).
\end{aligned}\right\} \quad (7)$$

In these equations γ is the ratio of heats, a_2 is the hydrodynamic sound speed downstream of the shock, and b_2, defined in (2*d*), is a magnetohydrodynamic sound speed downstream of the shock. It is important to note that all of the quantities (6) have fixed sign except M_u and that the last three quantities (7) can also change sign.

The slope of the integral curve at 2 is then given by

$$\left.\begin{aligned}
&\frac{T - T_2}{u - u_2} = \frac{\Delta T}{\Delta u} = \frac{N_u}{k\kappa - N_T}, \quad \frac{B - B_2}{w - w_2} = \frac{\Delta B}{\Delta w} = \frac{\eta\kappa - L_w}{L_B},\\
&\frac{\Delta w}{\Delta u} = \frac{L_B J_u}{(J_B - \lambda\kappa)(L_w - \eta\kappa) - L_B J_w},\\
&\frac{\Delta B}{\Delta u} = \frac{J_u (L_w - \eta\kappa)}{L_B J_w - (J_B - \lambda\kappa)(L_w - \eta\kappa)} = \frac{M_T N_u - (N_T - k\kappa)(M_u - \eta'' u_2 \kappa)}{M_B (N_T - k\kappa)}.
\end{aligned}\right\} \quad (8)$$

κ is required to be negative in order that $x \to +\infty$ as point 2 is approached. (It is easy to show that the integral curves leave 1 at $x = -\infty$.) In the limiting cases to be studied, this requirement on κ indicates immediately the conditions for the existence of a boundary layer. Furthermore, the form of κ together with (8) indicates the scale factors for the various quantities.

In the previous section, where the case $\eta, \lambda \ll \eta'', k$ was briefly considered, it was found that if η and λ were simultaneously set equal to zero a discontinuity must occur. This conclusion is readily checked here by setting $\eta = \lambda = 0$ in (5). Using the first equation of (7), the result is $\kappa \doteqdot N_T/k = mC_v/k$. Since κ is positive the integral curve will not approach 2 as $x \to +\infty$ and a discontinuity must be inserted.

4. Solutions for some limiting cases

The simplified equations for several limiting cases will now be derived. In nearly all cases the solution of these equations involves simple quadrature and the solutions are not carried out explicitly since their general nature is clear.

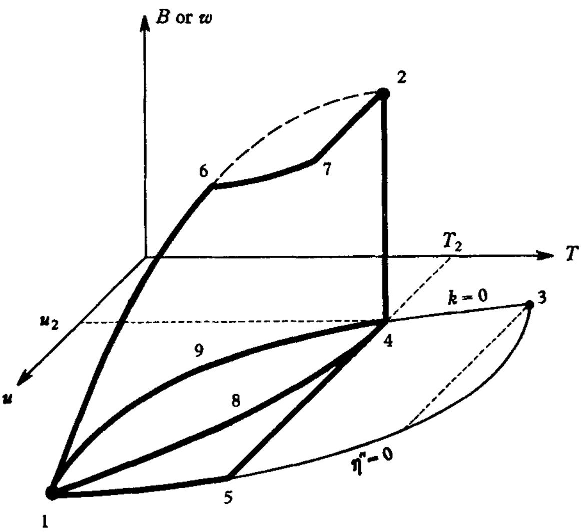

FIGURE 2. Integral curves in the (B, u, T) and (w, u, T) spaces.

The integral curves for many of these limiting cases are readily sketched in one or the other of the three-dimensional phase spaces corresponding to the reduced systems (3) and (4). This will become clear when some limiting cases are actually considered. For the moment, it is useful to examine the surfaces in the (B, u, T) space obtained by setting λ, η'', and k equal to zero in system (3) and the surfaces in the (w, u, T) space obtained by setting η, η'', and k equal to zero in system (4). Because of the similarity between systems (3) and (4), the following discussion applies to either but, for discussion purposes, only the (B, u, T) space corresponding to system (3) will be considered.

Some of the features to be discussed now are shown in figures 2 and 3. Points 1 and 2 are the singular points corresponding to the initial and final conditions, respectively. From equation $(3a)$ it is clear that the surface corresponding to

$\lambda = 0$ consists of the two planes $B = 0$ and $u = u_2$. The curves in the $B = 0$ plane labelled $\eta'' = 0$ (1–5–3) and $k = 0$ (1–4–3) are the intersections of the $B = 0$ plane with the $\eta'' = 0$ and $k = 0$ surfaces, respectively. These curves correspond to the so-called 'non-viscous' and 'non-conducting' curves in hydrodynamic shock theory [see, for example, figure $D5a$ in Hayes (1958)]; points 1 and 3 are the initial and final end-points, respectively, for a hydrodynamic shock. The intersection of the $\eta'' = 0$ and $k = 0$ surfaces is shown schematically as the dashed curve

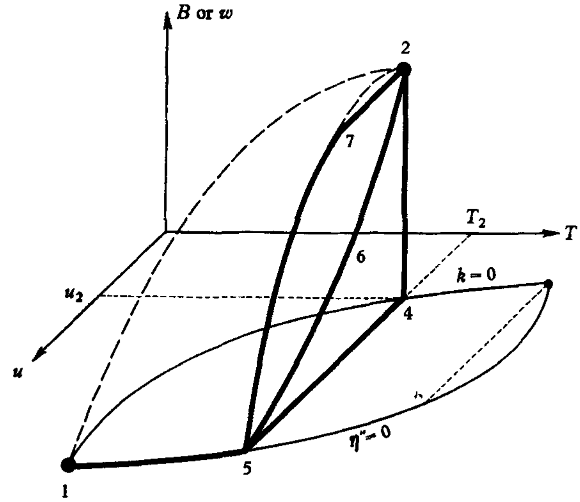

FIGURE 3. Integral curves in the (B, u, T) and (w, u, T) spaces.

from 1 to 2 in figure 3 (1–6–2 in figure 2). There is, of course, a continuation of this curve from 2 to 3. An important property of the $k = 0$ surface that should be noted is that the straight line $u = u_2$, $T = T_2$ (4–2) lies in this surface; only B and w vary along this line. These surfaces play important roles in many of the limiting cases to be studied.

(A) $\eta, \eta'', k \ll \lambda$ (*low electrical conductivity*)

Although this limiting case was treated in detail by Bleviss (1959), it is repeated here to illustrate the present method. Setting $\eta = \eta'' = k = 0$ in (5), the result for κ is

$$\kappa \doteqdot -\frac{J_u M_B N_T}{\lambda(N_T M_u - N_u M_T)} = -\frac{b_2^2 u_2}{\lambda(u_2^2 - a_2^2)} = O\left(\frac{1}{\lambda}\right), \tag{9}$$

where $(2d)$ and the first two equations of (7) have been used. Since κ is negative only when $u_2 > a_2$, the procedure of setting $\eta = \eta'' = k = 0$ leads to a uniformly valid limiting solution only when $u_2 > a_2$.

The above equation will now be used to illustrate several additional important conclusions that can always be drawn from the result for κ. The dissipation mechanism is always clear from the diffusivity contained in the result. In the present case λ is the diffusivity and Joule heating is the dissipation mechanism. Since the thickness of a one-dimensional shock wave is always proportional to the diffusivities involved, the scale factor for x is immediately apparent. Note that this is consistent with the fact that κ has the dimensions of reciprocal length and is always essentially proportional to the reciprocal of a diffusivity.

Using the fact that $\kappa = O(1/\lambda)$, the following conclusions can be drawn from (8)

$$\left.\begin{aligned} \frac{T-T_2}{u-u_2} = \frac{\Delta T}{\Delta u} = O(1), \quad \frac{\Delta w}{\Delta u} = O(1), \\ \frac{\Delta B}{\Delta w} = O(1), \qquad\qquad \frac{\Delta B}{\Delta u} = O(1). \end{aligned}\right\} \tag{10}$$

With ΔB, Δw, Δu and ΔT all $O(1)$ and with the x-range over which these variables change appreciably (i.e. the shock thickness) also $O(1)$ since $\lambda = O(1)$, the approximate equations for this limiting case are obtained from (1) by simply setting $\eta = \eta'' = k = 0$. This corresponds to setting $\eta'' = k = 0$ in (3). This simplified system of equations can be reduced to a single first-order ordinary differential equation that is readily solved.

The integral curve in the (B, u, T) space is the curve 1–6–2 (figure 2), i.e. the intersection of the $\eta'' = 0$ and $k = 0$ surfaces. The requirement $u_2 > a_2$ restricts the results to those curves for which B increases monotonically, i.e. curves for which B is single-valued between 1 and 2.

$u_2 < a_2$

When $u_2 < a_2$, κ cannot vary as $1/\lambda$ but must instead be a function of one or more of the diffusivities η, η'' and k. This means that the solution obtained by setting $\eta = \eta'' = k = 0$ must be discontinuously adjusted through a boundary layer to the final conditions at 2. The nature and location of this boundary layer will now be determined.

Using the fact that $\lambda\kappa \gg 1$, (5) reduces to

$$(L_w - \eta\kappa)\,[(N_T - k\kappa)\,(M_u - \eta'' u_2\,\kappa) - N_u M_T] \doteqdot 0.$$

The first factor cannot be zero since then κ would not be negative. Therefore, κ is given by the quadratic equation

$$(N_T - k\kappa)\,(M_u - \eta'' u_2\,\kappa) - N_u M_T \doteqdot 0. \tag{11}$$

With $N_T M_u - N_u M_T < 0$, this equation yields negative values for κ. Note that since η is not contained in the result for κ the dissipation in this boundary layer must be due to viscosity (η'') and heat conduction (k) only, just as in the case of the hydrodynamic shock. Then the essential results for this boundary layer will be independent of how η is related to η'' and k. Some minor results are affected by this relationship and, to avoid writing out all the cases, the realistic assumption $\eta = O(\eta'')$ is made.

Assuming η'' and k to be of the same order and writing $k = \alpha\eta''$ [with $\alpha = O(1)$] in (11), it is clear that $\kappa = O(1/\eta'')$. Using this result, (8) becomes

$$\left.\begin{aligned} \frac{\Delta T}{\Delta u} = O(1), \quad \frac{\Delta w}{\Delta u} = O\left(\frac{\eta''}{\lambda}\right), \\ \frac{\Delta B}{\Delta w} = O(1), \quad \frac{\Delta B}{\Delta u} = O\left(\frac{\eta''}{\lambda}\right). \end{aligned}\right\} \tag{12}$$

This shows that to first order w and B are constant through this boundary layer, having their downstream values w_2 and B_2.

With the scale factors known from the result for κ and (12), the following $O(1)$ quantities for this boundary layer can now be defined

$$T^* = T - T_2 = \Delta T, \quad u^* = u - u_2 = \Delta u, \quad w^* = \frac{\lambda}{\eta''}\Delta w, \quad B^* = \frac{\lambda}{\eta''}\Delta B, \quad x^* = \frac{x}{\eta''}.$$

Subsituting these new variables into (1) and carrying only the largest terms in each equation, the first-order equations for the boundary layer are

$$\frac{dB^*}{dx^*} = u^* B_2, \tag{13a}$$

$$\frac{dw^*}{dx^*} = mw^* - \frac{B_0 B^*}{\mu}, \tag{13b}$$

$$u\frac{du^*}{dx^*} = mRT^* + m\left(u - \frac{RT_2}{u_2}\right)u^*, \tag{13c}$$

$$\alpha\frac{dT^*}{dx^*} = mC_v T^* + \frac{mRT_2}{u_2}u^* - \frac{m}{2}u^{*2}. \tag{13d}$$

Note that (13*c*) and (13*d*) can be obtained from (1*c*) and (1*d*) by setting $w = w_2$ and $B = B_2$. Then (13*c*) and (13*d*) are the equations for a hydrodynamic shock and must be solved simultaneously for $u^*(x^*)$ and $T^*(x^*)$. Knowing $u^*(x^*)$, $B^*(x^*)$ can be obtained from (13*a*) and then $w^*(x^*)$ can be obtained from (13*b*). With $B^*(x^*)$ a known function, it is easy to show that the solution of (13*b*) is

$$mw^* = \frac{B_0 B^*}{\mu}. \tag{13b'}$$

(13*a*) and (13*b*′) arise because $B^*(x^*)$ and $w^*(x^*)$ have slope discontinuities across the boundary layer.

The x position of the boundary layer is determined by the criterion already discussed by Bleviss (1959). Briefly, this can be described by examination of the typical shock profiles $u(x)$ and $B(x)$ shown in figure 4. The curves *abcd* are the profiles obtained by setting $\eta = \eta'' = k = 0$ in (1). The boundary layer occurs at b, where $B = B_2$, and the solid curves show the correct single-valued profiles.

The integral curve in the (B, u, T) space is indicated in figure 2. The curve 1–6–2 intersects the plane $B = B_2$ at 6 and at 2. The correct integral curve is then the solid curve from 1 to 6 followed by a curve from 6 to 2 that lies in the plane $B = B_2$. The portion of the curve from 6 to 2 is the integral curve for a hydrodynamic shock between 6 and 2 and the properties of this curve are well known. Note that x is constant along the curve from 6 to 2. The solid curve 6–7–2 corresponds to the special case treated below.

(B) $\eta, \eta'' \ll k \ll \lambda$

This is a special case of (A) in which $\eta = O(\eta'')$ but the Prandtl number is small. It is clear that the foregoing results are altered only for the case $u_2 < a_2$ and that only the portion of the integral curve from 6 to 2 is affected. Since this is a hydrodynamic shock for zero Prandtl number the results are well known.

Assuming $\kappa = O(1/k)$ and using the fact that $\eta''\kappa \ll 1$, (11) reduces to

$$\kappa \doteqdot \frac{N_T M_u - N_u M_T}{k M_u} = \frac{m C_v (u_2^2 - a_2^2)}{k(u_2^2 - a_2^2/\gamma)}. \tag{14}$$

This is negative for $a_2^2/\gamma < u_2^2 < a_2^2$. For this case it is easily shown that the resulting equations for u and T are those for a hydrodynamic shock with $\eta'' = 0$, i.e.

$$R(T - T_2) + \left(u - \frac{RT_2}{u_2}\right)(u - u_2) = 0, \tag{15a}$$

$$k\frac{dT}{dx} = mC_v(T - T_2) + \frac{mRT_2}{u_2}(u - u_2) - \frac{m}{2}(u - u_2)^2. \tag{15b}$$

In figure 2 the integral curve from 6 to 2 is the intersection of the plane $B = B_2$ with the surface $\eta'' = 0$. The condition $a_2^2/\gamma < u_2^2 < a_2^2$ restricts this case to those curves for which T increases monotonically from 6 to 2. Then, along the integral curve the dissipation mechanisms are Joule heating between 1 and 6 and heat conduction between 6 and 2. The case shown in figure 2 is discussed below.

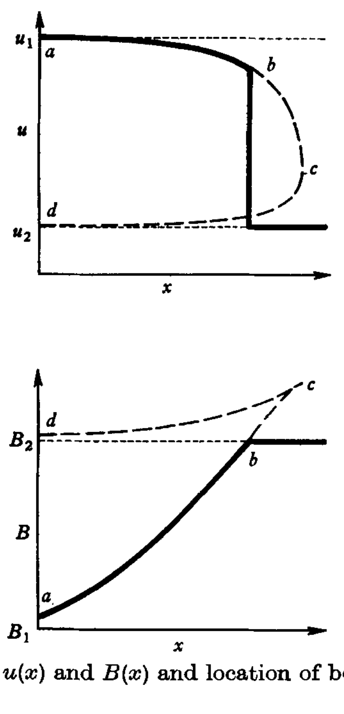

FIGURE 4. Shock profiles $u(x)$ and $B(x)$ and location of boundary layer for $u_2 < a_2$.

$u_2^2 < a_2^2/\gamma$

To obtain a negative value for κ, it is necessary to take $\kappa = O(1/\eta'')$ in (11). The result is

$$\kappa \doteqdot \frac{M_u}{\eta'' u_2} = \frac{m(u_2^2 - a_2^2/\gamma)}{\eta'' u_2^2}. \tag{16}$$

There is now a second boundary layer within the first that discontinuously adjusts the solution for the first boundary layer to the final conditions at 2. This case corresponds to that described by Bleviss (1959) as a 'shock within a shock within a shock'. Using (8), the slope of the integral curve at 2 is given by

$$\left.\begin{aligned} \frac{\Delta T}{\Delta u} &= O\left(\frac{\eta''}{k}\right), & \frac{\Delta w}{\Delta u} &= O\left(\frac{\eta''}{\lambda}\right), \\ \frac{\Delta B}{\Delta w} &= O(1), & \frac{\Delta B}{\Delta u} &= O\left(\frac{\eta''}{\lambda}\right), \end{aligned}\right\} \tag{17}$$

For this second boundary layer only u changes, the other quantities being constant at their final values. Defining the $O(1)$ quantities

$$u^* = \Delta u, \quad T^* = \frac{k}{\eta''}\Delta T, \quad w^* = \frac{\lambda}{\eta''}\Delta w, \quad B^* = \frac{\lambda}{\eta''}\Delta B, \quad x^* = \frac{x}{\eta''},$$

the equations for the second boundary layer are

$$\frac{dB^*}{dx^*} = u^* B_2, \tag{18a}$$

$$\frac{dw^*}{dx^*} = mw^* - \frac{B_0 B^*}{\mu}, \tag{18b}$$

$$u\frac{du^*}{dx^*} = m\left(u - \frac{RT_2}{u_2}\right)u^*, \tag{18c}$$

$$\frac{dT^*}{dx^*} = \frac{mRT_2}{u_2}u^* - \frac{m}{2}u^{*2}, \tag{18d}$$

where (18*b*) simplifies in the same way as (13*b*). u is the only quantity that is varying and it is determined by (18*c*), which is readily solved. (18*c*) is obtained from (1*c*) by setting $T = T_2$ and $B = B_2$. The integral curve is shown in figure 2 as 1–6–7–2. The curve from 6 to 7 is the same as that discussed just above but in this case the curve does not increase monotonically to 2; it intersects the plane $T = T_2$ at 7. The portion of the curve from 7 to 2 is parallel to the u-axis. Along the integral curve the dissipation mechanisms are Joule heating between 1 and 6, heat conduction between 6 and 7, and viscosity (η'') between 7 and 2.

(C) $\lambda \ll \eta'' \ll k \ll \eta$ (*high electrical conductivity*)

When λ is much less than the other diffusivities the case will be referred to as one of high electrical conductivity. This case is essentially case (B) with η and λ interchanged, and it is being presented to show the similarities of and the differences between the two cases. Only the main results will be given. It will be found that there are comparable integral curves in the (B, u, T) and (w, u, T) spaces for the two cases, but that the conditions for the appearances of the boundary layers and the equations governing the shock structure are quite different.

$u_2^2 > a_2^2 + b_2^2$

When $u_2^2 > a_2^2 + b_2^2$, $\kappa = O(1/\eta)$ and the governing equations are obtained from (1) by setting $\lambda = \eta'' = k = 0$. The integral curve in the (w, u, T) space is 1–6–2 in figure 2 with w increasing monotonically from 1 to 2. The dissipation is due to viscosity (η).

$a_2^2/\gamma + b_2^2 < u_2^2 < a_2^2 + b_2^2$

When $u_2^2 < a_2^2 + b_2^2$ a boundary layer arises regardless of the relative magnitudes of λ, η'' and k. This boundary layer is not a hydrodynamic shock but is, instead, the parallel shock studied by Marshall, i.e. a shock through which the flow is not turned but the magnetic field parallel to the shock front varies. For this shock, B_0 does not interact with the flowing gas. The integral curve goes from 1 to 6 and then from 6 to 2 in the plane $w = w_2$. Note that B varies from 6 to 2.

With $\eta'' \ll k$, a single boundary layer arises if $a_2^2/\gamma + b_2^2 < u_2^2$. The integral curve from 6 to 2 is the intersection of the $w = w_2$ plane and the $\eta'' = 0$ surface with T increasing monotonically. The dissipation mechanism is heat conduction. The equations governing the changes in B, u and T are (1a), (1c) and (1d) with $\lambda = \eta'' = 0$ and $w = w_2$.

$u_2^2 < a_2^2/\gamma + b_2^2$

A second boundary layer now arises in which the dissipation is due to viscosity (η'') and through which $T = T_2$. The integral curve is 1–6–7–2. The equations for B and u are (1a) and (1c) with $\lambda = 0$, $w = w_2$ and $T = T_2$.

(D) $\eta \ll \lambda \ll \eta'', k$

Although unrealistic, all cases where η and λ are small compared with the other diffusivities have a number of interesting features. Since η and λ cannot be simultaneously set equal to zero, this case will always involve at least one boundary layer. As previously discussed, the integral curve leaves 1 in the plane $B = 0$ along *some* integral curve for the hydrodynamic shock and this part of the integral curve must terminate in the $u = u_2$ plane. The exact terminus is determined by a study of the boundary layer which adjusts the solution to the conditions at 2.

If η is set equal to zero and it is assumed that $\kappa = O(1/\lambda)$ in (5), a negative result cannot be obtained for κ. However, if $\eta = 0$ and it is assumed only that κ contains λ in such a way that $\eta''\kappa \gg 1$, (5) reduces to

$$(N_T - k\kappa)(u_2\lambda\eta''\kappa^2 - J_u M_B) \doteqdot 0.$$

Only the second parenthesis leads to $\kappa < 0$, with the result

$$\kappa \doteqdot -\sqrt{\frac{J_u M_B}{\lambda\eta'' u_2}} = -\frac{B_2}{\sqrt{\lambda\eta''\mu}}. \tag{19}$$

This result is independent of the order of magnitude of $k\kappa$ and of the relative orders of magnitude of k and η''. Note that $k\kappa$ can be $\ll 1$, $O(1)$, or $\gg 1$ when $k \ll \eta''$ and that $k\kappa \gg 1$ when $k = O(\eta'')$ or $k \gg \eta''$. The essential results are the same for all of these cases and, to avoid writing them all out, it will be assumed that $k\kappa \gg 1$. Then, from (8),

$$\left.\begin{aligned} \frac{\Delta T}{\Delta u} &= O\left(\frac{1}{k\kappa}\right) = O\left(\sqrt{\frac{\lambda\eta''}{k^2}}\right), & \frac{\Delta w}{\Delta u} &= O\left(\frac{1}{\lambda\kappa}\right) = O\left(\sqrt{\frac{\eta''}{\lambda}}\right), \\ \frac{\Delta B}{\Delta w} &= O(1), & \frac{\Delta T}{\Delta w} &= O\left(\frac{\lambda}{k}\right). \end{aligned}\right\} \tag{20}$$

The integral curve in the (B, u, T) space (figure 2) is now clear. $u = u_2$ and $T = T_2$ through the boundary layer. Then this portion of the integral curve is 4–2, the intersection of the $u = u_2$ and $T = T_2$ planes, and only B (and w) varies along this portion. The portion of the integral curve that lies in the $B = 0$ plane must extend from 1 to 4. For $\eta'' = O(k)$, a typical integral curve is 1–8–4–2. The curve 1–8–4 is parallel to the u-axis at 4, i.e. $\partial T/\partial u = 0$.

The curve 1–8–4 is obtained by solving (1 c) and (1 d) with $w = B = 0$. When $k \ll \eta''$ the integral curve is 1–9–4–2 and the equations for 1–9–4 are the same as for 1–8–4 but with $k = 0$. When $\eta'' \ll k$ the integral curve is 1–5–4–2. Now, two boundary layers occur, the first from 5 to 4 and the second from 4 to 2. The equations for the portion 1–5 are (1 c) and (1 d) with $\eta'' = w = B = 0$. The equation for the portion 5–4 is (1 c) with $T = T_2$ and $B = 0$.

The detailed characteristics of the boundary layer 4–2 will now be studied. The following $O(1)$ quantities are defined:

$$B^* = \Delta B, \quad w^* = \Delta w, \quad u^* = \sqrt{\frac{\eta''}{\lambda}}\Delta u, \quad T^* = \frac{k}{\lambda}\Delta T, \quad x^* = \frac{x}{\sqrt{(\lambda\eta'')}}. \tag{21}$$

The orders of magnitude of the time rate of energy dissipation per unit volume for the four dissipation mechanisms are

$$\eta''\left(\frac{du}{dx}\right)^2 = \frac{1}{\eta''}\left(\frac{du^*}{dx^*}\right)^2 = O\left(\frac{1}{\eta''}\right), \quad \frac{\lambda}{\mu}\left(\frac{dB}{dx}\right)^2 = O\left(\frac{1}{\eta''}\right),$$

$$\eta\left(\frac{dw}{dx}\right)^2 = O\left(\frac{\eta}{\lambda\eta''}\right), \quad \frac{k}{T}\left(\frac{dT}{dx}\right)^2 = O\left(\frac{\lambda}{k\eta''}\right).$$

This shows that the dissipation is due to Joule heating and viscosity (η'').

The order of magnitude of the entropy increase across the boundary layer can be obtained by integrating the time rate of entropy production per unit volume across the boundary layer

$$\int \frac{\eta''}{T}\left(\frac{du}{dx}\right)^2 dx = \int \sqrt{\frac{\lambda}{\eta''}}\left(\frac{du^*}{dx^*}\right)^2 \frac{dx^*}{T} = O\left(\sqrt{\frac{\lambda}{\eta''}}\right).$$

This means that there is no entropy increase across this boundary layer to first order! This result can also be derived from the fact that u and T and, hence, ρ and p are constant through the boundary layer.

Using (21), the system (1) reduces to

$$\frac{dB^*}{dx^*} = u^*B, \tag{22a}$$

$$mw^* - \frac{B_0 B^*}{\mu} = mw - \frac{B_0 B}{\mu} = 0, \tag{22b}$$

$$\frac{du^*}{dx^*} = \frac{B^2 - B_2^2}{2\mu}, \tag{22c}$$

$$\frac{dT^*}{dx^*} = u^*\left(\frac{mRT_2}{u_2} - \frac{B^2 - B_2^2}{2\mu}\right). \tag{22d}$$

Even though $u = u_2$, essentially, in the boundary layer, u is still coupled to B through the pair of equations (22a) and (22c). Once this pair of equations is solved for $u^*(x^*)$ and $B^*(x^*)$, $w^*(x^*)$ and $T^*(x^*)$ are obtained from (22b) and (22d).

The solution of (22a) and (22c) can be reduced to a single quadrature in the following way. It is convenient to work with the variable B^2 instead of B^*. Dividing one equation by the other and integrating leads to the result

$$u^* = \frac{1}{\sqrt{2\mu}}\sqrt{\left[B_2^2 \ln\left(\frac{B_2^2}{B^2}\right) + B^2 - B_2^2\right]}.$$

If this is substituted into (22a), $B(x^*)$ is given implicitly by

$$x^* = \sqrt{\frac{\mu}{2}} \int \frac{dB^2}{B^2\sqrt{\left[B_2^2 \ln\left(\frac{B_2^2}{B^2}\right) + B^2 - B_2^2\right]}} + \text{const.},$$

where the constant determines the arbitrary origin of x^*.

If η and λ are interchanged the results are essentially the same, the roles of B and w being interchanged.

(E) $\lambda \ll \eta$, $\eta'' \ll k$

This is the physically realistic case of high electrical conductivity. Again, since λ and η cannot both be set equal to zero, a boundary layer will always occur. Then it is expected that κ will contain both η and η''.

Setting $\lambda = 0$ and letting $\eta\kappa = O(1)$, $\eta''\kappa = O(1)$, and $k\kappa \gg 1$, (5) reduces to

$$\eta\kappa J_B(M_u - \eta'' u_2 \kappa) + J_u M_B (L_w - \eta\kappa) \doteqdot 0. \tag{23}$$

This quadratic equation yields negative values for κ without further restrictions.

Writing $\eta'' = \alpha\eta$ [with $\alpha = O(1)$] in (23), it is clear that $\kappa = O(1/\eta)$. Then (8) becomes

$$\left.\begin{aligned} &\frac{\Delta T}{\Delta u} = O\left(\frac{1}{k\kappa}\right) = O\left(\frac{\eta}{k}\right), \quad &&\frac{\Delta w}{\Delta u} = O(1), \\ &\frac{\Delta B}{\Delta w} = O(1), &&\frac{\Delta B}{\Delta u} = O(1). \end{aligned}\right\} \tag{24}$$

The temperature is constant through this boundary layer at its final value T_2.

Carrying through the analysis, it is easy to show that the equations governing the variations of B, w and u are (1a), (1b) and (1c) with $\lambda = 0$ and $T = T_2$. Using system (4), the equations for w and u can be written

$$\eta \frac{u}{m} \frac{dw}{dx} = (u - u_2)\, w, \tag{25a}$$

$$\eta'' \frac{u^2}{m} \frac{du}{dx} = u\left(u - \frac{RT_2}{u_2}\right)(u - u_2) + \frac{u_2}{2}\left(w^2 - \frac{u^2}{u_2^2} w_2^2\right), \tag{25b}$$

with B given by

$$B = \frac{w}{u} B_0. \tag{25c}$$

From (25a) and (25b) it is readily deduced that the upstream and downstream end-points in the (w, u, T) space (figure 3) are 5 and 2, respectively. A typical integral curve for this boundary layer is 5–6–2, where the curve lies in the $T = T_2$ plane. Since $\eta'' \ll k$, the portion of the integral curve from 1 to 5 lies along the intersection of the $w = 0$ plane and the $\eta'' = 0$ surface. Along the integral curve the dissipation mechanisms are heat conduction between 1 and 5 and viscosity (η and η'') between 5 and 2.

(F) $\lambda \ll \eta \ll \eta'' \ll k$

This is a subcase of (E) in which $\eta \ll \eta''$. Using (23) the result for κ is

$$\kappa \doteqdot -\sqrt{\frac{J_u M_B L_w}{\eta\eta'' u_2 J_B}} = -B_2\sqrt{\frac{m}{\eta\eta''\mu u_2}},$$

and it is clear that this is just case (D) with $\eta'' \ll k$ and η and λ interchanged. Then the integral curve in the (w, u, T) space of figure 3 is 1–5–4–2 with the dissipation due to heat conduction between 1 and 5, viscosity (η'') between 5 and 4, and viscosity (η and η'') between 4 and 2. As before, there is no entropy increase between 4 and 2.

(G) $\lambda \ll \eta'' \ll \eta \ll k$

This is a subcase of (E) in which $\eta'' \ll \eta$. Assuming $\kappa = O(1/\eta)$ and setting $\eta'' = 0$, (23) reduces to

$$\kappa \doteqdot -\frac{J_u M_B L_w}{\eta(J_B M_u - J_u M_B)} = -\frac{m b_2^2}{\eta(u_2^2 - a_2^2/\gamma - b_2^2)}. \tag{26}$$

This is negative for $u_2^2 > a_2^2/\gamma + b_2^2$. The equations for this case are given by the system (25) with $\eta'' = 0$. The integral curve is 1–5–7–2 following the dashed curve from 7 to 2. The portion 5–7–2 is the intersection of the $T = T_2$ plane and the $\eta'' = 0$ surface. The condition $u_2^2 > a_2^2/\gamma + b_2^2$ restricts the portion 5–7–2 to curves for which w increases monotonically from 5 to 2. The dissipation along 5–7–2 is due to viscosity (η).

$u_2^2 < a_2^2/\gamma + b_2^2$

Assuming $\kappa = O(1/\eta'')$, (23) reduces to

$$\kappa \doteqdot \frac{J_B M_u - J_u M_B}{\eta'' u_2 J_B} = \frac{m(u_2^2 - a_2^2/\gamma - b_2^2)}{\eta'' u_2^2}. \tag{27}$$

A second boundary layer is now obtained. The integral curve is the heavy curve 1–5–7–2, with 7–2 along the intersection of the $w = w_2$ and $T = T_2$ planes. The equation for u along the portion 7–2 is (25*b*) with $w = w_2$ and the dissipation is due to viscosity (η'').

(H) $\eta, \eta'' \ll \lambda \ll k$

This is a physically realistic case with intermediate electrical conductivity. It can be shown that this case is very similar to that of (G) but with η and λ interchanged. For $u_2^2 > a_2^2/\gamma$ the integral curve in the (B, u, T) space of figure 3 is again 1–5–7–2, following the dashed curve from 7 to 2. For $u_2^2 < a_2^2/\gamma$ the integral curve is the heavy curve 1–5–7–2 and the dissipation mechanisms are heat conduction between 1 and 5, Joule heating between 5 and 7, and viscosity (η'') between 7 and 2.

(I) $\eta'' \ll k \ll \lambda \ll \eta$

In all the cases that have been studied, there have been at most two boundary layers. With four diffusivity parameters, it would be expected that for some order-of-magnitude orderings there would be three boundary layers, the maximum number possible. The present physically unrealistic case is being presented to illustrate one such case. Only a brief summary of the results will be given.

$u_2^2 > a_2^2 + b_2^2$

The results here are the same as for case (C) when $u_2^2 > a_2^2 + b_2^2$.

$a_2^2 < u_2^2 < a_2^2 + b_2^2$

For this condition the first boundary layer occurs. w is constant and equal to w_2 through this boundary layer. A plot of $w(x)$, using the equations for $u_2^2 > a_2^2 + b_2^2$, would be similar to that shown for $B(x)$ in figure 4 and, similarly, the location of the boundary layer is determined by the condition $w = w_2$. The dissipation in this boundary layer is due to Joule heating and the equations governing the variations in B, u and T are (1a), (1c) and (1d) with $w = w_2$ and $k = \eta'' = 0$.

$a_2^2/\gamma < u_2^2 < a_2^2$

Now, a second boundary layer arises. In general, when $u_2 < a_2$ this boundary layer is simply a hydrodynamic shock, regardless of the relative magnitudes of η'' and k. With $\eta'' \ll k$, the condition $u_2^2 > a_2^2/\gamma$ guarantees that T increases monotonically through the boundary layer. $B = B_2$ (and $w = w_2$) through this boundary layer and the location of the second boundary layer within the first is determined by this condition. The dissipation is due to heat conduction and the equations for u and T are obtained from (1c) and (1d) by setting $w = w_2$, $B = B_2$ and $\eta'' = 0$.

$u_2^2 < a_2^2/\gamma$

For this condition, a third boundary layer occurs. Only the velocity u varies through this boundary layer and the dissipation is due to viscosity (η''). The equation for u is (1c) with $B = B_2$ and $T = T_2$.

(J) $\eta, \eta'' \ll k, \lambda$

This case and the following one complete the list of physically realistic cases given in § 2. The results will be summarized very briefly.

The integral curve in the (B, u, T) space is a curve between 1 and 2 that lies in the $\eta'' = 0$ surface and is bracketed by the curves 1–6–2 (figure 2) and 1–5–7–2 (figure 3). This case reduces to (B) when $k \ll \lambda$ and reduces to (H) when $\lambda \ll k$. The system of equations for this case is (1) with $\eta = \eta'' = 0$ or the reduced system (3) with $\eta'' = 0$. The conditions for the appearance of boundary layers are more complicated than for cases (B) and (H) and will depend upon the relative magnitudes of k and λ. These conditions are determined from the quadratic equation for κ obtained by setting $\eta = \eta'' = 0$ in (5).

(K) $\lambda, \eta, \eta'' \ll k$

Since λ and η cannot be set equal to zero simultaneously, a boundary layer must occur. Upstream of the boundary layer the integral curve is 1–5 in either the (B, u, T) or (w, u, T) space. This curve is a portion of an integral curve for a hydrodynamic shock and the equations that govern the variations of u and T along this curve are (1c) and (1d) with $\eta'' = w = B = 0$, the dissipation mechanism being heat conduction. The boundary layer occurs between 5 and 2, the temperature being constant at its final value T_2. The integral curve for the boundary layer cannot be sketched in the (B, u, T) or (w, u, T) space since λ and η are of the same order of magnitude. The equations that govern the variations of B, w and u through the boundary layer are (1a), (1b) and (1c) with $T = T_2$.

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